Growth and the Size of the R&D Sector in a Model with Endogenous Fertility*

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Abstract
We examine the relationship between the intensity of the research activity and the long-run level of economic growth in a model with endogenous technical progress and endogenous choice of fertility. To conduct the analysis, we introduce a policy tool charged on labor employed by the research sector. When the size of the R&D sector is modified, we show that the level of growth can increase, decrease, or does not vary.

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The outcome depends on two factors: the weight that individuals place on offspring relative to their level of consumption; and the value of the elasticity of the share of working labor devoted to research with respect to policy changes.

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## 1. Introduction

The literature on growth yields ambiguous conclusions on the relationship between the size of the research and development (R&D) sector and per-capita long-term economic growth. In the basic R&D-based literature (e.g. Romer, 1990; Grossman and Helpman, 1991; Aghion and Howitt, 1992), the usual result is that an increase of the amount of labor devoted to research leads to faster long-term growth. This result, however, has been challenged in the semi-endogenous growth literature (e.g. Jones, 1995; Kortum, 1997; Segerstrom, 1998) in which the size of the R&D sector does not affect long-term growth. This is because in these models the level of growth is determined by the population growth rate which is taken as an exogenous variable. Finally, using a R&D-based model with endogenous population growth, Jones (2003) has shown that growth can be negatively correlated with the size of the R&D sector. To explain his result, which can appear to be counter-intuitive, he argues that a larger research sector takes labor from the (alternative) use of producing offspring. As population growth is the ultimate engine of growth, this implies that a greater amount of resources devoted to research reduces long-term growth.

In the present article, we attempt to shed a new light on the above issues, i.e. on the relationship between the size of the R&D sector and per-capita long-term economic growth. We show, in a single framework, that the three kinds of outcomes derived in the literature are indeed possible and we determine the conditions under which they occur. For this purpose, we develop a slightly modified version
of the Jones’ (2003) model in which population growth is endogenous and technical progress is the outcome of a costly innovative activity performed by firms (see below and Section 2).¹

To analyze the relationship between the size of the R&D sector and long-term growth, we assume the government’s intervention which can use a policy instrument to influence the size of the research sector. Like Jones (2003), we show that such policy can affect long-term growth through an indirect mechanism. Basically, changes in research intensity alter growth through their effects on the choice of fertility of individuals. However, we show that parents can choose to bring up fewer, more or even an unchanged number of children. Thus, long-term growth can be lower, higher or unchanged implying that it can be positively, negatively or uncorrelated with the size of the R&D sector. We show that the outcome depends on two factors: the weight that individuals place on offspring relative to consumption and the value of the elasticity of the share of working labor devoted to research with respect to policy changes.

It is worthwhile to note that one novelty of our framework concerns the manner the R&D activity is funded. In contrast with the basic R&D literature which considers that pieces of knowledge (i.e. non rival or non-depletable goods according to the textbook definition, for instance that of Mas-Colell, Whinston and Green, 1995, ch. 11) are freely available and cannot be protected by patents (because only private goods can be), in our framework we assume that pieces of knowledge are directly protected by patents. Thereby, users of knowledge must pay innovators to have the right to use it. This assumption can be seen as a formalization of ideas that have been developed for years by various authors such as Arrow (1962), Scotchmer (1991), Dasgupta et alii (1996), Gallini and Scotchmer (2003) but which have not found their way in growth models.

¹ Therefore, this paper can also be seen as an answer to Segerstrom (1998) who suggests to analyze the robustness of the result of Jones (2003).
In addition to being technically simple, we think that our formalization accounts for the recent evolution of intellectual property law. Indeed, since the mid-eighties we can observe that pieces of knowledge are directly patented. It is now possible to obtain patents for databases, software, business plans (see Scotchmer, 1999, for more details). To emphasize the distinctive properties of these goods, Quah (1997, 2001) labels them as “knowledge-products” because their properties resemble to those of knowledge. It is possible to argue that “knowledge products” must be embedded in private goods to be used as formalized in the basic R&D-based literature. However, in the case of new technologies, the “private” goods have an almost nonexistent marginal cost of production (CD-ROM) which can even be zero (an on-line copy). It is therefore akin to a public good and almost consubstantial with the piece of knowledge itself. We then think that it is interesting to state the problem as if knowledge, i.e. the public good, is directly patented.2

Knowledge being a public good, there are difficulties of funding associated with research in a decentralized economy. Formally, there are two types of problems. The first ones are standard in economics literature. They are related to the possibility of verifying which agents use knowledge; they are linked to the possibility of excluding agents that do not pay to use knowledge; they concern the problems of information on the marginal profitability of knowledge for an agent, e.g. the willingness to pay of this agent to use knowledge. These problems prevent innovators from appropriating the entire amount of the surplus they create.

A second type of problem comes from the non-convexity of technologies using knowledge as a productive factor. The replication argument states that there are constant returns to scale with respect to private inputs and increasing returns to scale with respect to both

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2 It is important to mention that our results do not depend on this formalization. The standard model in which innovators are granted a monopoly on the production and the sale of private goods that embody new pieces of knowledge would yield similar results.
private and public inputs. In a competitive market the payment of private factors fully exhausts revenues. Thus, firms are unable to pay for the public good they use.\(^3\) To solve this problem of existence, either we must assume that knowledge is publicly funded which is not realistic, or we must consider an equilibrium with imperfect competition. In this paper, we construct a dynamic general equilibrium with Cournot competition and free entry. Thereby, firms get funds to buy knowledge and research is privately funded.

The remainder of the paper is organized as follows: in Section 2, we present the model. In Sections 3, we characterize the equilibrium of the model and discuss its properties regarding the relationship between the size of the R&D sector and long-term growth. We conclude in Section 4.

2. The Model

We build on previous work by Smulders and Van de Klundert (1995, 1997), Peretto (1998, 1999a, 1999b) and Jones (2003). We consider a model in continuous time populated by a representative dynastic family which has \(L_t\) identical members. Each individual is endowed with one unit of time that she allocates between working and non-working activities. Non-working activities consist in bringing up children to adulthood. Working activities consist in the production of differentiated consumption goods and in the production of new pieces of knowledge through R&D activities. We assume that differentiated goods are produced by an exogenous number of sectors \((j = 1, \ldots, N)\), each one comprising \(Q_{j}^{t}\) identical firms \((q_{j}^{t} = 1, \ldots, Q_{j}^{t})\). It should be noted that the number of firms \(Q_{j}^{t}\) is an endogenous variable whose value, assumed to be continuous, will be determined in equilibrium (see Section 3). Formally, in sector \(j\), firm \(q_{j}^{t}\) produces a quantity \(X_{q_{j}^{t}}\) of differentiated good \(j\), with the technology

\(^3\) See for instance Kaizuka (1965), Sandmo (1972), Manning et al. (1985), Feehan (1989), Romer (1990), Jones (2003) for more details on this point.
where $0 < \sigma < 1$, $L_{qjt}^X$, is the quantity of labor employed and $A_t$ is the total stock of knowledge in the economy.

Simultaneously, firms conduct R&D to produce new pieces of knowledge. This is a simplification, although such behavior can be found in several sectors such as the pharmaceutical industry, software or aeronautics. Let us note that a continuum of pieces of knowledge (i.e. information goods or knowledge products) constitutes the total stock of knowledge at every point in time $t$. A piece of knowledge is an indivisible, infinitely-lived, differentiated, public good. It can be a scientific report, a database, or a software algorithm. Formally, it is a point of the segment $[0, A_t]$. Denoting by $A_{qjt}$ the stock of knowledge produced by firm $q_j$ until date $t$, we have $A_t = \sum_{j=1}^{N} \sum_{q_j=1}^{Q_{jt}} A_{qjt}$.

Following Jones (1995, 2003), we assume that the R&D technology of firm $q_j$ is given by

$$\dot{A}_{qj} = \xi_t \left( L_{qjt}^A \right) \left( A_t \right)^{\phi},$$

where $L_{qjt}^A$ is the quantity of labor employed to conduct research and $\phi < 1$ allows past knowledge to either increase ($\phi > 0$), or decrease ($\phi < 0$), current research productivity. The parameter $\xi_t$ is taken as given by each firm. It measures external effects in research due to duplication or redundancy in some research projects. It satisfies $\xi_t = \delta \left( L_t^A \right)^{\chi-1}$, where $\delta > 0$, $L_t^A = \sum_{j=1}^{N} \sum_{q_j=1}^{Q_{jt}} L_{qjt}^A$, is the aggregate quantity of labor devoted to research, and $0 < \chi \leq 1$. If $\chi = 1$, externalities caused by the duplication effect vanish.

Following Barro and Becker (1988, 1989), Barro and Sala-i-Martin (2004, ch. 9), we assume that the members of the representative family are linked to each other by altruism. Preferences of the head of
the dynastic family are represented by\(^4\)

\[
U = \int_0^\infty e^{-\rho t} \left\{ \ln \left( \sum_{j=1}^N (c_{jt})^{\alpha} \right)^{1/\alpha} + \theta \ln L_t \right\} dt,
\]

where \(0 < \alpha < 1\), \(c_{jt}\) is the per-capita purchase of each differentiated good \(j\), \(\rho > 0\) is the rate of time preferences and \(L_t\) is the size of the family which is also the total number of individuals. Parameter, \(\theta > 0\), is the weight that the head of the dynastic family places on offspring relative to consumption. If \(\theta = 1\), (which is the case analyzed by Jones, 2003), the head of the dynastic family is indifferent, in terms of utility gains, between an increase of one percent of consumption and an increase of one percent of the size of the family. If \(\theta < 1\) (resp. \(\theta > 1\)), she prefers, in terms of utility gains, an increase of one percent of consumption (resp. of the size of the family), to an increase of one percent of the size of the family (resp. consumption). We will see in Section 3 that this parameter plays an important role for the results we obtain.

Population evolves through time according to

\[
\dot{L}_t = (n_t - m) L_t,
\]

where \(n_t\) is the fertility choice of an individual and \(m, m > 0\), is the constant and exogenous mortality rate. Becker (1991) points out that rearing children to adulthood is costly, especially in the mother’s time in societies in which women are the primary providers of child care. Following the standard literature on growth and fertility, we assume that each individual spends a fixed amount of time to bring up \(n_t\) children. Denoting by \(L^n_t\) the total quantity of labor devoted to child bearing and given that individuals are identical, the per-capita share of

\(^4\) In chapter 9 of their book, Barro and Sala-i-Martin (2004) show that the continuous version of the utility function given in the text (see equation (3)) is equivalent to the one we obtain with a discrete time OLG framework in which parents are altruistic to their children. That is to say, their level of utility depends on their own level of consumption but also on the level of utility attained by their children.
labor devoted to this activity is given by \( L^n_t / L_t \). Like Jones (2003), we assume that the technology of production of children is given by

\[
\frac{L^n_t}{L_t} = B\left(n_t\right),
\]

where \( B(\cdot) \) is a continuous, strictly increasing and strictly convex function. It verifies \( B(0) = 0 \) and \( B(b) = 1 \) where \( b, b > m \), represents the maximum number of children that a parent can have: if the entire amount of the labor endowment of an individual is devoted to bring up children (i.e. \( L^n_t / L_t = 1 \)), the maximum value of \( n_t \) is \( b \), i.e. we have \( 0 \leq n_t \leq b \). The property \( B(0) = 0 \) means that the number of children that a parent can have is zero if she does not devote any time to child bearing. To fix notation, let us define \( B_n(\cdot) \) and \( B_{nn}(\cdot) \) as the first and second derivatives of \( B(\cdot) \). To simplify the analysis, we assume that \( B_n(m)/[1-B(m)] < \theta/\rho \), and \( \partial\{B_{nn}(n)/[1-B(n)]\} + \{B_n(n)/[1-B(n)]\}^2 \}/\partial n > 0 \), are always satisfied. These conditions ensure that we have an interior solution which exists and is unique (i.e. we focus on a solution for which the fertility rate verifies \( 0 < n_t < b \) in equilibrium).

At each moment, individuals allocate their labor endowment between working activities (production of the differentiated consumption goods and research) and child bearing. The aggregate labor constraint is

\[
L_t = L^X_t + L^A_t + L^n_t,
\]

where \( L^X_t = \sum_{j=1}^N \sum_{q_j=1}^{Q_{jt}} L^X_{q_jt} \).

In the rest of the paper, we denote by \( s_t = L^A_t / (L^X_t + L^A_t) \) the share of working time devoted to research and by \( (1 - s_t) = L^X_t / (L^X_t + L^A_t) \) the share of working time devoted to the production of differentiated goods. Note that in contrast with Jones (2003) who assumes that the fraction of working time devoted to research, \( s_t \), is exogenously given, in this paper this variable is determined endogenously. As for the weight individuals place on offspring relative to consump-
tion, \( \theta \), we will see in Section 3 that the variable \( s_i \) plays an important role for the results we obtain.

Finally, since the whole amount of every differentiated good is consumed, we have

\[
L_t c_{jt} = X_{jt},
\]

where \( X_{jt} = \sum_{q_{jt}=1}^{Q_{jt}} X_{q_{jt}} \).

3. Dynamic General Equilibrium with Cournot Competition and Free Entry

In this section, we aim to study the effects on the long-term growth rate of a policy tool that modifies the size of the R&D sector. To this end, we assume the government’s intervention by means of a tax \( \tau_t \) charged on the amount of labor employed in research. That is, on each unit of labor employed to conduct research, firm \( q_j \) must pay \( \tau_t w_t L^A_{q_{jt}} \), where \( w_t \) is the wage per unit of labor, which adds to the usual cost of labor given by \( w_t L^A_{q_{jt}} \). By changing the level of the tax \( \tau_t \), the government modifies the size of the R&D sector, which in turn can modify the long-run growth rate. For simplicity, we assume that the proceeds from the tax are redistributed to the representative family through a lump-sum transfer, \( T_t \). Assuming that the budget constraint of the government is balanced at each moment, we have: \( T_t = \tau_t w_t L^A_t \) for all \( t \).

In what follows, we construct a dynamic general equilibrium with Cournot competition and free entry in which knowledge is directly priced and research is privately funded. Following Grimaud and Tournemaine (2007), we assume that the \( N \) markets of differentiated goods \( (X_{jt}) \) are imperfectly competitive, i.e. the \( Q_{jt} \) firms of each sector \( j \) (\( j = 1, \ldots, N \)) compete à la Cournot. By selling differentiated
goods at price $p_{jt}$ which is greater than the marginal cost of production, firms get resources that allow them to buy knowledge. Then, assuming free entry on each differentiated good market ensures that profits are zero. Note that the number of firms, $Q_{jt}$, which composes each differentiated sector $j$, is determined by using the free entry condition. Moreover, in this equilibrium the payment of knowledge appears as a fixed cost for each firm.

For knowledge, we assume that it is traded using bilateral contracts between inventors and users. To keep the analysis simple, and because it does not yield any new insights for the purpose of the paper, we assume that sellers are able to extract the whole willingness to pay of all buyers, i.e. there are no problem of verification, exclusion and information.\footnote{It is possible to extend the analysis to the case in which sellers extract only a fraction of the willingnesses to pay due to problems of verification, exclusion and information. See, for instance, Grimaud and Tournemaine (2006) for a discussion of this issue.} The price a firm is willing to pay to use a piece of knowledge at date $t$ is determined by the marginal profitability of this piece of knowledge. Denoting by $\tilde{\pi}_{q_jt}$ the profit of firm $q_j$ before the payment of knowledge, the price of a piece of knowledge is given by $v_{q_jt} = \frac{\partial \tilde{\pi}_{q_jt}}{\partial A_t}$. It follows that the price paid by firm $q_j$ to use a piece of knowledge from $t$ to infinity is given by $V_{q_jt} = \int_t^{\infty} v_{q_jt} e^{-\int_t^s r_u du} ds$, where $r_u$ denotes the interest rate; and the payment perceived by any firm for the sale of a piece of knowledge, i.e. the value of a piece of knowledge, is given by $V_t = \int_t^{\infty} v_s e^{-\int_t^s r_u du} ds$, where $V_t = \sum_{j=1}^{N} \sum_{q_j=1}^{Q_{jt}} V_{q_jt}$ and $v_s = \sum_{j=1}^{N} \sum_{q_j=1}^{Q_{jt}} v_{q_j,s}$. Differentiating the expression of $V_t$ with respect to time yields the usual condition: $r_t = v_t/V_t + g_V$, where $g_V$ denotes the growth rate of any variable $z$.

Finally, we assume that the labor market whose price is normalized to one ($w_t = 1$) and the financial market are perfectly competitive. Formally, an equilibrium is defined as follows:
**Definition 1:** An equilibrium with Cournot competition is a set of profiles of number of firms \(\{Q_{jt}\}, j=1,...,N\), of quantities of differentiated goods \(\{X_{qjt}\}, q_{j}=1,...,Q_{jt}, j=1,...,N\), of quantities of knowledge \(\{A_{qjt}\}, q_{j}=1,...,Q_{jt}, j=1,...,N\), of quantities of labor \(\{L^X_{qjt}\}, \{L^A_{qjt}\}, q_j=1,...,Q_{jt}, j=1,...,N, \{L^i\}\) and of prices \(\{v_{qjt}\}, \{Q_{qjt}\}, q_{j}=1,...,Q_{jt}, j=1,...,N, \{p_{jt}\}, j=1,...,N, \{r_{j}\}\) such that:

- the head of the dynastic family maximizes utility;
- firms maximize profits;
- the labor market and the financial market are perfectly competitive and clear;
- on each differentiated good market, there is Cournot competition with free entry;
- pieces of knowledge are traded using bilateral contracts.

### 3.1 Behavior of Agents

#### 3.1.1 Individuals

The head of the dynastic family maximizes (3) subject to (4)-(6) and the budget constraint

\[ \dot{D}_t = r_t D_t + w_t (L_t^X + L_t^A) - L_t \sum_{j=1}^{N} p_{jt} c_{jt} + T_t \]

(recall that \(w_t\) is equal to one), where \(D_t\) is the stock of wealth of the dynastic family. Since knowledge, \(A_t\), is the only asset of the firms, we have:

\[ D_t = A_t V_t. \]

After substitution, the current-value Hamiltonian of this problem is:

\[ CVH = \ln \left[ \sum_{j=1}^{N} \left( c_{jt} \right)^{1/\alpha} \right] + \theta \ln L_t + \lambda_t \left[ r_t D_t + w_t (1 - B(n_t)) L_t - L_t \sum_{j=1}^{N} P_{jt} c_{jt} + T_t \right] + \zeta_t (n_t - m) L_t, \]

where \( \lambda_t \) and \( \zeta_t \) are co-state variables.

The first order conditions are:

\[ \frac{\partial CVH}{\partial c_{jt}} = 0, \quad \frac{\partial CVH}{\partial n_t} = 0, \]

\[ \frac{\partial CVH}{\partial D_t} = -\dot{\lambda}_t + \rho \lambda_t, \quad \frac{\partial CVH}{\partial L_t} = -\dot{\zeta}_t + \rho \zeta_t. \]

After some manipulation, we get respectively:
Moreover, the transversality conditions are:
\[
\frac{\lambda_t}{\zeta_t} B_n(n_t) = \zeta_t \, , \tag{8}
\]
and
\[
r_t + \frac{\dot{\lambda}_t}{\lambda_t} = \rho \, , \tag{9}
\]

Using the resource constraint, \(\theta = \frac{\lambda_t}{\zeta_t} \left[ (1 - B(n_t)) - \sum_{j=1}^{N} p_{jt} c_{jt} \right] + (n_t - m) + \frac{\zeta_t}{\zeta_t} = \rho \). \(\tag{10}\)

Moreover, the transversality conditions are: \(\lim_{t \to \infty} \lambda_t D_t e^{-\rho t} = 0\), and \(\lim_{t \to \infty} \zeta_t L_t e^{-\rho t} \).

Standard manipulation of (8) yields the aggregate demand function for each consumption good \(j\):
\[c_{jt} = E_t \left( p_{jt} \right)^{\frac{1}{\alpha - 1}} , \text{ where } E_t = \frac{\sum_{j=1}^{N} p_{jt} c_{jt}}{\sum_{j=1}^{N} \left( p_{jt} \right)^{\frac{1}{\alpha - 1}}} . \]

Then, using the resource constraint, \(L_t c_{jt} = X_{jt}\) (see equation (7)), we get the inverse demand function of differentiated good \(j\):
\[
p_{jt} = \left( L_t E_t \right)^{1-\alpha} \left( \sum_{q=1}^{Q_{jt}} X_{q,j} \right)^{\alpha - 1} . \tag{11}\]

Differentiating (8) with respect to time and combining the result with (10) leads to the usual Keynes-Ramsey rule:
\[
r_t = (1 - \alpha) g_{c_{jt}} + g_{\Omega_t} + (n_t - m) + g_{p_{jt}} + \rho , \tag{13}\]

where \(g_{\Omega_t}\) is the growth rate of \(\Omega_t = \sum_{j=1}^{N} \left( c_{jt} \right)_{t}^{\alpha} \).

Finally, manipulations of (8), (9), (10) and (11) yields
which is an arbitrage condition between fertility (i.e. a larger size of family in the future) and investments in research (i.e. more consumption in the future).

### 3.1.2 Firms

Firms have two activities: 1) They produce and sell differentiated goods on an imperfect market (competition à la Cournot); 2) simultaneously, they produce and sell pieces of knowledge that are traded using bilateral contracts between inventors and users. At each moment, firm \( q_j \) maximizes

\[
\tilde{\pi}_{q,j} = p_{jt} X_{q,j,t} - L_{q,j,t}^X + V_t A_{q,j,t} - \left(1 + \tau_t \right) L_{q,j,t}^A
\]

(profit without payment of knowledge), subject to

\[
X_{q,j,t} = \left( A_t \right)^\sigma \left( L_{q,j,t}^A \right) \phi
\]

(see equation (1)),

\[
\dot{A}_{q,j,t} = \xi_t \left( L_{q,j,t}^A \right) \phi
\]

(see equation (2)) and the inverse demand function

\[
p_{jt} = \left( L_t E_t \right)^{1-\alpha} \left( \sum_{q_j=1}^{Q_t} X_{q,j,t} \right)^{\alpha - 1}
\]

(see equation (12)),

where the term \( E_t \) is taken as given because each firm \( q_j \) is assumed to be “atomistic”. After substitutions, we get the following program:

\[
\text{Max} \tilde{\pi}_{q,j} = X_{q,j} \left[ \left( L_t E_t \right)^{1-\alpha} \left( \sum_{q_j=1}^{Q_t} X_{q,j} \right)^{\alpha - 1} - \left( A_t \right)^\sigma \right] + V_t \xi_t \left( L_{q,j,t}^A \right) \phi - \left(1 + \tau_t \right) L_{q,j,t}^A.
\]

The first order condition with respect to \( X_{q,j,t} \) leads to

\[
X_{jt} = L_t E_t \left\{ \left( A_t \right)^\sigma \left[ 1 + \left( \alpha - 1 \right) \frac{X_{q,j,t}}{X_{jt}} \right] \right\}^{1/(1-\alpha)}.
\]
The first order condition with respect to \( L_{q,t}^A \) yields:

\[
V_t \xi_t (A_t)^\phi = 1 + \tau_t .
\]  

(17)

The willingness to pay at date \( t \) to use a piece of knowledge at \( t \) is 

\[ v_{q,t} = \frac{\partial \pi_{q,t}}{\partial A_t} . \]

Using (1) and (17), we get:

\[
v_{q,t} = \frac{\partial \pi_{q,t}}{\partial A_t} = \frac{\sigma L^X_{q,t} + \phi (1 + \tau_t) L_{q,t}^A}{A_t} .
\]  

(18)

Therefore, the willingness to pay of firm \( q, j \) to use a piece of knowledge from \( t \) to infinity is:

\[
V_{q,t} = \int_t^\infty \frac{\sigma L^X_{q,s} + \phi (1 + \tau_s) L_{q,s}^A}{A_s} e^{-\int_t^s \tau}\, ds .
\]  

(19)

Note that the term \( v_{q,t} \) is composed of two parts. The first one, \( \sigma L^X_{q,t} / A_t \), can be interpreted as the willingness to pay at date \( t \) to use a piece of knowledge at \( t \) for the production of the differentiated good; the second one, \( \phi (1 + \tau_t) L_{q,t}^A / A_t \), is the willingness to pay to use a piece of knowledge at date \( t \) in order to make research at \( t \). That is, we recover the public good nature of knowledge inside the firm: each unit of knowledge is used twice by each firm.

Finally, the free entry condition on the markets of differentiated goods is

\[ \pi_{q,t} = \tilde{\pi}_{q,t} - V_{q,t} \dot{A}_t = 0, \forall q, j, \forall j, \]

(20)

where \( V_{q,t} \dot{A}_t \) represents the payment for new pieces of knowledge.
3.2 Symmetric Equilibrium and Characterization of the Steady-state

Since the model contains two state variables whose initial values are historically given \((L_0, A_0)\), it contains transitional dynamics. However, as well as being analytically difficult, we believe that such an exercise will not greatly contribute to our main objectives. Thus, we do not treat this question and we focus on the symmetric equilibrium at steady-state.

**Definition 2:** A symmetric equilibrium is characterized by a number of firms in each sector \(j\), quantities and prices that are identical for all \(q_j\) and for all \(j\): \(Q_j = Q_t\) for all \(j\), \(X_{q,t} = X_t / Q_t\), \(L_{q,t}^X = L_t^X / Q_t\), \(L_{q,t}^L / (NQ_t)\), \(L_{q,t}^A = L_t^A / Q_t = L_t^A / (NQ_t)\), \(A_{q,t} = A_t / Q_t = A_t / (NQ_t)\), for all \(q_j\) and for all \(j\); \(p_{jt} = p_t\), \(v_{qjt} = v_{jt} / Q_t = v_t / (NQ_t)\), \(V_{qjt} = V_{jt} / Q_t = V_t / (NQ_t)\) for all \(q_j\) and for all \(j\).

Proposition 1 summarizes the results we get:

**Proposition 1:** At steady-state, under the assumptions \(B_n(m) /[1-B(m)] < \theta / \rho\), and \(\partial \{B_{nn}(n) /[1-B(n)] +\{B_n(n) /[1-B(n)]\}^2\}/\partial n > 0\), the symmetric general dynamic equilibrium with Cournot competition and free entry is characterized by a unique and constant fertility rate \(n\) whose value is solution of

\[
\begin{align*}
  f(n) &= h(n, \tau), \\
  \text{with } f(n) &= B_n(n) /[1-B(n)] \quad \text{and } h(n, \tau) = \theta / \rho + \tau(\theta - 1)s(n, \tau) / \rho. \\
  \text{The share of working labor devoted to research, } s(n, \tau) &= L_t^A / (L_t^X + L_t^A), \\
  \text{is given by } &
\end{align*}
\]

\[
  s = s(n, \tau) = \frac{\sigma g_A / (1 + \tau)}{(1-\phi) g_A + \rho + \sigma g_A / (1 + \tau)}. 
\]

The number of firms is given by:

\[
  Q = (1 - \alpha) \left\lfloor 1 + \frac{1 - s}{s(1 + \tau)} \right\rfloor. 
\]

The growth rates of quantities are:
The prices are

\[ p_t = \left( A_t \right)^\sigma \left[ 1 + \left( \alpha - 1 \right)/Q \right]^{-1}, \quad (27) \]

\[ v_t = \frac{\sigma L_t^Y + \phi(1 + \tau)A_t^A}{A_t}, \quad V_t = \frac{(1 + \tau)}{\delta(L_t^A)^{\tau - 1}(A_t)^{\delta}}, \quad (28) \]

\[ r = (n - m) + \rho. \quad (29) \]

**Proof:** See Appendix. ■

From Proposition 1, the steady-state value of the fertility rate is implicitly determined by equation (21) which corresponds to the arbitrage condition between fertility and research investments (14) under symmetry at steady-state. Figures 1, 2 and 3 give a graphical representation of this solution for \( \theta < 1 \), \( \theta = 1 \) and \( \theta > 1 \), respectively. As shown in Appendix, the function \( h(n, \tau) \) can be decreasing (\( \theta < 1 \)), constant (\( \theta = 1 \)) or increasing (\( \theta > 1 \)) with respect to \( n \).

The main implication of Proposition 1 is that the choice of fertility of individuals depends on the level of the tax rate, \( \tau \), through the function \( h(n, \tau) \). Since there is a positive linear relationship between growth and fertility (see equations (24) and (26)), this implies that changes in the degree of research intensity (via a change of \( \tau \)), can affect the level of the growth rate. This issue is addressed in more details in the next sub-section.\(^6\)

\(^6\) Of course, policy changes affect also the equilibrium level of prices and the number of firms, \( Q \), in each sector. For instance, for given \( A_t \), a higher level of tax induces a higher price for innovations, \( V_t \). The effect on the interest rate, on the price of differentiated goods and on the number of firms is ambiguous. It depends on how the policy change affects the allocation of working labor between the production of differentiated goods and research, and/or on the choice of fertility of individuals. We do not go further in the comments as we think these questions go beyond the scope of this paper.
Figure 1 $\theta < 1$

Figure 2 $\theta = 1$
3.3 Effects of Policy Changes

In this Section, we study the consequences of the variation of the tax rate, $\tau$, on the steady-state equilibrium. We assume that the economy is initially on the balanced growth path, and suddenly the government decides to implement an unanticipated, permanent, marginal increase in $\tau$. Such policy leads to an increase of the cost of labor allocated to research (see equations (15) and (17)), thus to a reduction of the amount of labor employed in research: the size of the R&D sector diminishes.

The effects on the fertility rate (i.e. on the growth rate) are more complex. As shown in Appendix, we have:

$$
\frac{dn}{d\tau} = \frac{s(n, \tau)}{\rho} \left(\frac{\theta - 1}{1 + \varepsilon(n, \tau)}\right),
$$

where $\varepsilon(n, \tau) = \tau \left[ \frac{\partial s(n, \tau)}{\partial \tau} \right] / s(n, \tau) < 0$ is the elasticity of the share of working labor devoted to R&D with respect to policy changes and $\frac{\partial f(n)}{\partial n} - \frac{\partial h(n, \tau)}{\partial n} > 0$.

Figure 3 $\theta > 1$
Equation (30) shows that the effects of an increase of the tax, $\tau$, on the fertility rate, $n$, depends on two elements: the weight that individuals place on their offspring relative to consumption, $\theta$, and the elasticity of the share of working labor devoted to R&D with respect to policy changes, $\varepsilon(n, \tau)$. Results are summarized in Table 1 where we present the signs of the derivatives of $n$ (row) with respect to $\tau$ in the cases $\theta<1$, $\theta=1$, $\theta>1$ (columns).

Table 1 Variation of the fertility rate, $n$, when the tax rate, $\tau$, increases

<table>
<thead>
<tr>
<th>$\theta&lt;1$</th>
<th>$\theta=1$</th>
<th>$\theta&gt;1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{dn}{d\tau}$</td>
<td>$&lt; 0$ if $\varepsilon(n, \tau)&gt;-1$</td>
<td>$&gt; 0$ if $\varepsilon(n, \tau)&gt;-1$</td>
</tr>
<tr>
<td>$= 0$ if $\varepsilon(n, \tau)=-1$</td>
<td>$= 0$ for all $\varepsilon(n, \tau)$</td>
<td>$= 0$ if $\varepsilon(n, \tau)=-1$</td>
</tr>
<tr>
<td>$&gt; 0$ if $\varepsilon(n, \tau)&lt;-1$</td>
<td>$&lt; 0$ if $\varepsilon(n, \tau)&lt;-1$</td>
<td></td>
</tr>
</tbody>
</table>

Table 1 shows that policy changes have ambiguous effects on the choice of fertility of individuals, thereby on the long-term growth rate. The interesting result is that the model encompasses all the different outcomes depicted in the standard literature regarding the relationship between the size of the R&D sector and the level of growth. Indeed, equation (30) has the following implications:

- If $\theta<1$ and $\varepsilon(n, \tau)>-1$ or $\theta>1$ and $\varepsilon(n, \tau)<-1$, an increase of the level of the tax, $\tau$, induces a decrease of the fertility rate. This case is consistent with the standard R&D-based literature (Romer, 1990; Grossman and Helpman, 1991; Aghion and Howitt, 1992) where a decrease of the size of the R&D sector goes along with a lower long-term growth rate.
- If $\theta=1$ and/or $\varepsilon(n, \tau)=-1$, policy changes have no effect on the fertility rate. This is the situation obtained in semi-endogenous growth models (Jones, 1995; Kortum, 1997; Segerstrom, 1998), in which economic policy cannot affect long-term economic growth.
• If \( \theta > 1 \) and \( \varepsilon(n, \tau) > -1 \) or \( \theta < 1 \) and \( \varepsilon(n, \tau) < -1 \), an increase in the level of the policy tool, \( \tau \), leads to an increase in the fertility rate. That is, we recover the counter-intuitive result of Jones (2003) stating that a decrease of the size of the R&D sector goes along with a greater long-term growth rate.

It is worthwhile to mention that a reduction of the size of the R&D sector leads to a lower rate of arrival of pieces of knowledge per unit of time (see equation (2)). In turn, this drives the future per-capita quantity of differentiated goods produced down (see equation (1)) implying a loss in terms of future consumption for individuals. Therefore, to understand the mechanisms that are behind the above results, it is necessary to analyze how individuals behave to try to compensate their loss in terms of future consumption.

Basically, when the share of working labor devoted to R&D with respect to policy changes is inelastic (\( \varepsilon(n, \tau) > -1 \)), the variation of the quantity of labor devoted to working activities is small. Thus, the loss in terms of future consumption is low. If individuals place a higher weight on consumption relative to offspring (\( \theta < 1 \)), they choose to work more and to bear fewer children because each additional unit of consumption gives them more utility than each new member in the family. There is an increase in the level of per-capita consumption, but only in the short-term. Indeed, since the quantity of labor devoted to children is lower, the fertility rate and thereby the long-term growth rate are lower. Graphically, there is a downward shift of the \( h(n, \tau) \) function in Figure 1. However, if individuals place a higher weight on offspring relative to their level of consumption (\( \theta > 1 \)), they prefer to bear more children because each new individual brings more utility than each new unit of consumption. Consequently, the long-term growth rate increases. Graphically, there is an upward shift of the \( h(n, \tau) \) function in Figure 3.

When the share of working labor devoted to R&D with respect to policy changes is elastic (\( \varepsilon(n, \tau) < -1 \)), the variation in the allocation of labor devoted to working activities is large inducing an important loss in terms of future consumption. If individuals place a higher
weight on consumption relative to offspring ($\theta<1$), they choose to bear more children because it means more individuals in the future and thus a higher level of growth. Graphically, there is an upward shift of the $h(n, \tau)$ function in Figure 1. However, if individuals place a higher weight on offspring relative to consumption ($\theta>1$), they try to compensate their loss by working more. Thus, they bear fewer children which leads to a lower long-term growth. Graphically, there is a downward shift of the $h(n, \tau)$ function in Figure 3.

If $\theta=1$ (i.e. individuals place an equal weight on consumption and on children) and/or $\theta=-1$, individuals do not modify the quantity of labor they allocate to bring up children. Thus, the fertility rate is unchanged. In this case, an increase in the quantity of labor devoted to the production of differentiated goods equal to the variation of the quantity of labor devoted to research compensates exactly the loss in terms of future consumption. This is the situation depicted in Figure 2 in which the $h(n, \tau)$ function is independent of the value of the policy tax, $\tau$.

4. Conclusion

In this paper, we have analyzed the relationship between the intensity of research activity and per-capita long-term growth. We have shown that the long-term growth rate can be positively, negatively or uncorrelated with the size of the R&D sector. This result reconciles all the results discussed in the relevant literature on growth. Our model shows that the outcome depends on two elements: the weight that individuals place on offspring relative to consumption and the value of the elasticity of the share of working labor devoted to research with respect to policy changes.

We think that the model we have developed accounts also for the recent evolution of intellectual property rights. In the new technology sector (software industry, biotechnology, etc.), pieces of knowledge are now directly protected by patents. We have tried to account for this evolution. We have considered an equilibrium in which knowledge is directly protected by patents so that any user must pay innovators to
have the right to use it. To deal with the problem of non-convexity, we have characterized a dynamic general equilibrium with Cournot competition and free entry. As firms compete in imperfectly competitive markets, they earn strictly positive profits that enable them to buy knowledge. Moreover, from a technical point of view, the methodology presented is very simple. Thus, we believe it could be used to study other questions such as sustainable development, credit funding, inequalities, unemployment, and so on.\(^7\)

Appendix

Characterization of the Symmetric Equilibrium

We proceed in two steps. First, we compute the main conditions that emerge at date \(t\) on each market. Second, we characterize the steady-state equilibrium.

1. Market Conditions for a Symmetric Equilibrium

Using (16), we get the equilibrium quantity of each differentiated good which is produced in each sector:

\[
X_t = L_t E_t \left\{ (A_t)^\sigma \left[ 1 + \frac{(\alpha - 1)}{Q_t} \right] \right\}^{1/(1-\alpha)}.
\]

Using (1), we get the aggregate production function of each differentiated good:

\[
X_t = \frac{(A_t)^\sigma L_t^X}{N}.
\]

Equation (12) yields the Cournot equilibrium price of differentiated good \(j\):

\(^7\) See for instance Grimaud and Tournemaine (2007) who use a similar model with endogenous human capital accumulation to study the relationship between growth and environmental policy.
Note that for $Q_t=1$ (monopoly case), we get the standard result $p_t = \left( \frac{\alpha A_t}{Q_t} \right)^\sigma$, where $\left( \frac{A_t}{Q_t} \right)^\sigma$ is the marginal cost and $1/\alpha > 1$ is the mark-up.

Combining the free entry condition (20) with equations (32) and (33),

$$\pi_{q,t} = \pi_t = \left[ 1 + \left( \frac{\alpha - 1}{Q_t} \right) \right]^{-1} L_t^X / (NQ_t) - L_t^X / (NQ_t) = (1 + \tau_t) L_t^A / (NQ_t),$$

we get , which gives:

$$\left[ 1 + \left( \frac{\alpha - 1}{Q_t} \right) \right]^{-1} L_t^X = \tau_t L_t^A + L_t^X + L_t^A. \quad (34)$$

From (18), we get the total willingness to pay at date $t$ to use a piece of knowledge at $t$ by all firms of the $N$ sectors:

$$\nu_t = \sum_{j=1}^N \sum_{q,j} \nu_{q,j} = \frac{\sigma L_t^X + \phi (1 + \tau_t) L_t^A}{A_t}. \quad (35)$$

From (17) and (19), the total payment perceived by a firm that sells a piece of knowledge at $t$ is

$$V_t = \int_t^\infty \frac{\sigma L_s^X + \phi (1 + \tau_s) L_s^A}{A_s} e^{-\int_t^s \xi_t(A_u) du} ds = \frac{(1 + \tau_t)}{\xi_t(A_t)^\phi}. \quad (36)$$

From (2), the law of motion of the total stock of knowledge is given by

$$\dot{A}_t = \delta \left( L_t^A \right)^{\xi_t \left( A_t \right)^\phi}, \quad (37)$$

where we recall that $\xi_t = \delta \left( L_t^A \right)^{\phi - 1}$. 


Using equation (13) with the assumption of symmetry, on gets the value of the interest rate:

\[ r_t = n_t - m + \rho = \frac{v_t}{V_t} + g_{V_t}. \]  

(38)

2. Steady-State

At steady-state, the growth rates, the shares of labor \((L^X_t / L_t, L^A_t / L_t, L^n_t / L_t)\) i.e. the fertility rate \(n\) and the number of firms in each sector are constant. Moreover, the existence of a balanced growth path requires the tax charged on labor devoted to research is constant: \(\tau_t = \tau\) for all \(t\).

Using (13), (14), (32), (33), we get

\[ \rho \frac{B_n(n)}{1 - B(n)} = 1 + (\theta - 1) \left[ 1 + \frac{\alpha - 1}{Q} \right]^{1 - s}, \]  

(39)

where we recall that \(s(n, \tau) = s = \frac{L^A_t}{(L^X_t + L^A_t)} = \frac{L^A_t}{[L^A_t - L_t B(n_t)]} \).

Using (35) and (36), we get \(v_t / V_t = \sigma(1 - s)g_A / [(1 + \tau)s] + \phi g_A\). Differentiating (33) and (36) with respect to time, we obtain \(g_p = -\sigma g_A\) and \(g_V = (1 - \chi)(n - m) - \phi g_A\). The aggregate technology of differentiated goods (32) implies \(g_c = g_A - (n - m) = \sigma g_A\). Finally from (37), we get \(g_A = \chi(n - m) / (1 - \phi)\). Therefore, using the labor constraint (6) with equation (38), we obtain

\[ (1 - \phi)g_A + \rho = g_A \frac{\sigma(1 - s)}{(1 + \tau)s}. \]  

(40)

The free entry condition in each sector requires
Equations (39), (40), (41) form a system of three equations with three unknowns: \( s, \frac{L^n}{L_t} = B(n), Q. \)

**Existence of a unique equilibrium fertility rate**

Recall that \( f(n) = B_n(n) / [1-B(n)] \) and \( h(n, \tau) = \theta/\rho + \tau(\theta-1)s(n, \tau)/\rho \), with \( s(n, \tau) \) given by equation (22), (see Proposition 1). We study the properties of \( f(n) \) and \( h(n, \tau) \).

Since \( B(\cdot) \) is strictly increasing and strictly convex, \( f(\cdot) \) is strictly increasing on the interval \([0,b]\). Since \( B(b) = 1 \), \( f(\cdot) \) admits a vertical asymptote for \( n = b \). Since \( g_A \) is an increasing function of \( n \), it is also the case for \( s(n, \tau) \). Moreover, we have \( \lim_{n \to m} s(n, \tau) \) and \( \lim_{n \to \infty} s(n, \tau) = \left[ \sigma / (1 + \tau) \right] / \left[ (1 - \phi) + \sigma / (1 + \tau) \right] \), that we call \( s(\infty, \tau) \).

Then, the properties of \( h(n, \tau) \) are the following: If \( \theta < 1 \) (resp. \( \theta > 1 \)), \( h(n, \tau) \) is a decreasing (resp. increasing) function of \( n \) on the interval \([m, \infty)\), such that \( h(m, \tau) = \theta/\rho \) and \( \lim_{n \to \infty} h(n, \tau) = \theta / \rho + \tau(1-\theta)s(\infty, \tau) \). If \( \theta = 1 \), \( h(n, \tau) = \theta / \rho \), for all \( n > 0 \).

Now, we can prove the existence of the uniqueness of the equilibrium fertility rate, \( n \). Looking at Figure 1 (\( \theta < 1 \)), Figure 2 (\( \theta = 1 \)) and Figure 3 (\( \theta > 1 \)), we see that, in order to have a unique intersection between \( f(\cdot) \) and \( h(\cdot) \), the two sufficient conditions are \( f(m) < \theta / \rho \), that is to say \( B_n(m) / [1-B(m)] < \theta / \rho \), and \( f'(n) > 0 \), that is to say \( \partial \{ B_n(n) / [1-B(n)] + \} / \partial n > 0 \). These two conditions are satisfied by assumption.

**Sign of \( d\tau/d\tau \)**

According to Proposition 1, under the assumptions \( B_n(m) / [1-B(m)] < \theta / \rho \) and \( \partial \{ B_n(n) / [1-B(n)] + \} / \partial n > 0 \), there exists a unique and constant fertility rate, \( n \), whose value is solution
of \( f(n) = h(n, \tau) \), where \( f(n) = B_n(n) / [1 - B(n)] \) and \( h(n, \tau) = \theta / \rho + \tau (\theta - 1) s(n, \tau) / \rho \). Let us define \( z(n, \tau) = f(n) - h(n, \tau) \). At steady-state, we have \( z(n, \tau) = 0 \). Applying the theorem of implicit functions to \( z(\cdot) \) along a balanced growth path, we get \( \frac{dn}{d\tau} = -\frac{\partial z(n, \tau)}{\partial n} = -\frac{\partial h(n, \tau)}{\partial n} + \frac{s(n, \tau)}{\rho} \). Therefore, we have

\[
\frac{dn}{d\tau} = \frac{s(n, \tau)}{\rho} \left( \theta - 1 \right) \frac{[1 + \varepsilon(n, \tau)]}{\partial f(n) / \partial n - \partial h(n, \tau) / \partial n}
\]

where \( \varepsilon(n, \tau) = [\partial s(n, \tau) / \partial \tau] / s(n, \tau) \). It is easy to verify that \( \partial s(n, \tau) / \partial \tau < 0 \) for all \( n \); moreover, Figures 1, 2, 3 imply that \( \partial f(n) / \partial n - \partial h(n, \tau) / \partial n > 0 \). Therefore, we deduce the results given in Table 1.

**References**


